

G. Λ_s -sets and G. V_s -sets*

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Abstract

In this paper we define the concepts of $g.\Lambda_s$ -sets and $g.V_s$ -sets and we use them in order to obtain new characterizations of semi- T_1 -, semi- R_0 - and semi- $T_{\frac{1}{2}}$ -spaces.

1 Introduction

Separation axioms stand among the most common and to a certain extent the most important and interesting concepts in Topology. One of the most well-known low separation axiom is the one which requires that singletons are closed, i.e. T_1 . In most studies, spaces under consideration are ‘by default’ T_1 .

In Digital Topology [8] several spaces that fail to be T_1 are important in the study of the geometric and topological properties of digital images [9, 10, 11]. Such is the case with the major building block of the digital n-space – the *digital line* or the so called *Khalimsky line*. This is the set of the integers, \mathbb{Z} , equipped with the topology \mathcal{K} , generated by $\mathcal{G}_{\mathcal{K}} = \{\{2n - 1, 2n, 2n + 1\} : n \in \mathbb{Z}\}$.

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Although the digital line is neither a T_1 -space nor an R_0 -space, it satisfies a couple of separation axioms which are a bit weaker than T_1 and R_0 , that is, the digital line is both a semi- T_1 -space and a semi- R_0 -space. This inclines to indicate that further knowledge of the behavior of topological spaces satisfying these two weak separation axioms (and some related ones) is required. This is indeed the intention of the present paper.

2 Preliminaries

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [12]. If (X, τ) is a topological space and $A \subseteq X$, then A is *semi-open* [12] if there exists $O \in \tau$ such that $O \subseteq A \subseteq \text{Cl}(O)$, where $\text{Cl}(O)$ denotes closure of O in (X, τ) . The complement A^c of a semi-open set A , is called *semi-closed* and the *semi-closure* of a set A denoted by $\text{sCl}(A)$, is the intersection of all semi-closed sets containing A .

The separation axioms R_0 was introduced by Davis in [4]. It requires that every open set contains the closures of its points. In 1975, Maheshwari and Prasad [14] introduced the class of semi- R_0 -spaces studied later by Di Maio [5] and by Janković and Reilly [7]. A topological space (X, τ) is called a *semi- R_0 -space* if every semi-open set contains the semi-closure of each of its singletons. One can easily observe that a space (X, τ) is a semi- R_0 -space if and only if every semi-open set is union of semi-closed sets. In this paper we focus our attention precisely on the sets which are union of semi-closed sets and study their basic properties. Although that the first impression might be that the separation axiom semi- R_0 is rather weak, one needs to consider the fact that T_4 -spaces, even ultraconnected spaces, need not be semi- R_0 (the easiest example is probably a Sierpinski space).

A generalized class of closed sets was considered by Maki in 1986 [15]. He investigated the sets that can be represented as union of closed sets and called them *V-sets*. Complements of *V-sets*, i.e., sets that are intersection of open sets are called *Λ -sets* [15]. In connection to semi- R_0 -spaces, observe that R_0 -spaces are precisely the spaces where the closed sets form a network for the topology, i.e., the spaces where every open set is a *V-set*. Every R_0 -space is a semi- R_0 -space [7] but not vice versa.

The family of all semi-open (resp. semi-closed) sets in (X, τ) will be denoted by $SO(X, \tau)$ (resp. $SC(X, \tau)$). In this note, we introduce and characterize the concepts of Λ_s -set, V_s -set, $g.\Lambda_s$ -set and $g.V_s$ -set in a topological space (X, τ) . In this article we give new characterizations of semi- $T_{\frac{1}{2}}$ -, semi- R_0 - and semi- T_1 -spaces in terms V_s - and $g.V_s$ -sets.

3 Λ_s -sets and V_s -sets

Definition 1 Let B be a subset of a topological space (X, τ) . We define the subsets B^{Λ_s} and B^{V_s} as follows:

$$B^{\Lambda_s} = \bigcap \{O / O \supseteq B, O \in SO(X, \tau)\} \text{ and } B^{V_s} = \bigcup \{F / F \subseteq B, F^c \in SO(X, \tau)\}.$$

In [6, 14], B^{Λ_s} is called the *semi-kernel* of B .

Proposition 3.1 Let A, B and $\{B_\lambda : \lambda \in \Omega\}$ be subsets of a topological space (X, τ) . Then the following properties are valid:

- (a) $B \subseteq B^{\Lambda_s}$;
- (b) If $A \subseteq B$, then $A^{\Lambda_s} \subseteq B^{\Lambda_s}$;
- (c) $B^{\Lambda_s \Lambda_s} = B^{\Lambda_s}$;
- (d) $[\bigcup_{\lambda \in \Omega} B_\lambda]^{\Lambda_s} = \bigcup_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$;
- (e) If $A \in SO(X, \tau)$, then $A = A^{\Lambda_s}$;
- (f) $(B^c)^{\Lambda_s} = (B^{V_s})^c$;
- (g) $B^{V_s} \subseteq B$;
- (h) If $B \in SC(X, \tau)$, then $B = B^{V_s}$;
- (i) $[\bigcap_{\lambda \in \Omega} B_\lambda]^{\Lambda_s} \subseteq \bigcap_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$;
- (j) $[\bigcup_{\lambda \in \Omega} B_\lambda]^{V_s} \supseteq \bigcup_{\lambda \in \Omega} B_\lambda^{V_s}$

Proof. (a) Clear by Definition 1.

(b) Suppose that $x \notin B^{\Lambda_s}$. Then there exists a subset $O \in SO(X, \tau)$ such that $O \supseteq B$ with $x \notin O$. Since $B \supseteq A$, then $x \notin A^{\Lambda_s}$ and thus $A^{\Lambda_s} \subseteq B^{\Lambda_s}$.

(c) Follows from (a) and Definition 1.

(d) Suppose that there exists a point x such that $x \notin [\bigcup_{\lambda \in \Omega} B_\lambda]^{\Lambda_s}$. Then, there exists a subset $O \in SO(X, \tau)$ such that $\bigcup_{\lambda \in \Omega} B_\lambda \subseteq O$ and $x \notin O$. Thus, for each $\lambda \in \Omega$ we have $x \notin B_\lambda^{\Lambda_s}$. This implies that $x \notin \bigcup_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$. Conversely, suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$. Then by Definition 1, there exist subsets $O_\lambda \in SO(X, \tau)$ (for all $\lambda \in \Omega$) such that $x \notin O_\lambda$, $B_\lambda \subseteq O_\lambda$. Let $O = \bigcup_{\lambda \in \Omega} O_\lambda$. Then we have that $x \notin \bigcup_{\lambda \in \Omega} O_\lambda$, $\bigcup_{\lambda \in \Omega} B_\lambda \subseteq O$ and $O \in SO(X, \tau)$. This implies that $x \notin [\bigcup_{\lambda \in \Omega} B_\lambda]^{\Lambda_s}$. Thus, the proof of (d) is completed.

(e) By Definition 1 and since $A \in SO(X, \tau)$, we have $A^{\Lambda_s} \subseteq A$. By (a) we have that $A^{\Lambda_s} = A$.

(f) $(B^{V_s})^c = \bigcap \{F^c / F^c \supseteq B^c, F^c \in SO(X, \tau)\} = (B^c)^{\Lambda_s}$.

(g) Clear by Definition 1.

(h) If $B \in SC(X, \tau)$, then $B^c \in SO(X, \tau)$. By (e) and (f): $B^c = (B^c)^{\Lambda_s} = (B^{V_s})^c$. Hence $B = B^{V_s}$.

(i) Suppose that there exists a point x such that $x \notin \bigcap_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$. Then, there exists $\lambda \in \Omega$ such that $x \notin B_\lambda^{\Lambda_s}$. Hence there exists $\lambda \in \Omega$ and $O \in SO(X, \tau)$ such that $O \supseteq B_\lambda$ and $x \notin O$. Thus $x \notin [\bigcap_{\lambda \in \Omega} B_\lambda]^{\Lambda_s}$.

(j) $[\bigcup_{\lambda \in \Omega} B_\lambda]^{V_s} = [((\bigcup_{\lambda \in \Omega} B_\lambda)^c)^{\Lambda_s}]^c = [(\bigcap_{\lambda \in \Omega} B_\lambda^c)^{\Lambda_s}]^c \supseteq [\bigcap_{\lambda \in \Omega} (B_\lambda^c)^{\Lambda_s}]^c = [\bigcap_{\lambda \in \Omega} (B_\lambda^{V_s})^c]^c = \bigcup_{\lambda \in \Omega} B_\lambda^{V_s}$
(by (f) and (h)). \square

Remark 3.2 In general $(B_1 \cap B_2)^{\Lambda_s} \neq B_1^{\Lambda_s} \cap B_2^{\Lambda_s}$, as the following example shows.

Example 3.3 Let (X, τ) be as in ([15, Example 2.9]) i.e., let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $B_1 = \{b\}$ and $B_2 = \{c\}$. Then, $(B_1 \cap B_2)^{\Lambda_s} = \emptyset$ but $B_1^{\Lambda_s} \cap B_2^{\Lambda_s} = \{b, c\}$.

Definition 2 In a topological space (X, τ) , a subset B is a Λ_s -set (resp. V_s -set) of (X, τ) if $B = B^{\Lambda_s}$ (resp. $B = B^{V_s}$).

Remark 3.4 By Proposition 3.1 (e) and (h) we have that:

- (a) If B is a Λ -set or if $B \in SO(X, \tau)$, then B is a Λ_s -set.
- (b) If B is a V -set or if $B \in SC(X, \tau)$, then B is a V_s -set.

Proposition 3.5 (a) The subsets \emptyset and X are Λ_s -sets and V_s -sets.

- (b) Every union of Λ_s -sets (resp. V_s -sets) is a Λ_s -set (resp. V_s -set).
- (c) Every intersection of Λ_s -sets (resp. V_s -sets) is a Λ_s -set (resp. V_s -set).
- (d) A subset B is a Λ_s -set if and only if B^c is a V_s -set.

Proof. (a) and (d) are obvious.

(b) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of Λ_s -set in a topological space (X, τ) . Then by Definition 2 and Proposition 3.1 (d), $\bigcup_{\lambda \in \Omega} B_\lambda = \bigcup_{\lambda \in \Omega} B_\lambda^{\Lambda_s} = [\bigcup_{\lambda \in \Omega} B_\lambda]^{\Lambda_s}$.

(c) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of Λ_s -set in (X, τ) . Then by Proposition 3.1 (h) and Definition 2 $[\bigcap_{\lambda \in \Omega} B_\lambda]^{\Lambda_s} \subseteq \bigcap_{\lambda \in \Omega} B_\lambda^{\Lambda_s} = \bigcap_{\lambda \in \Omega} B_\lambda$. Hence by Proposition 3.1 (a) $\bigcap_{\lambda \in \Omega} B_\lambda = [\bigcap_{\lambda \in \Omega} B_\lambda]^{\Lambda_s}$. \square

Recall that a space topological (X, τ) is called a *semi- T_1 -space* [13] if to each pair of distinct points x, y of (X, τ) there corresponds a semi-open set A containing x but not y and a semi-open set B containing y but not x , or equivalently, (X, τ) is a semi- T_1 -space if and only if every singleton is semi-closed.

Example 3.6 The digital line $(\mathbb{Z}, \mathcal{K})$ is an example of a semi- T_1 space and a semi- R_0 -space which is neither T_1 nor R_0 . Since all even singletons are closed, they are trivially semi-closed. On the other hand the odd integers are regular open (but not closed) and hence semi-closed too. Thus, $(\mathbb{Z}, \mathcal{K})$ is semi- T_1 and every semi-open sets is the union of all of its semi-closed singletons. So, the digital line is a semi- R_0 -space. On the other hand the isolated points in the digital line (i.e., the odd integers) can not be expressed as union of closed sets, which implies that the digital line is not an R_0 -space.

Proposition 3.7 A topological space (X, τ) is a semi- T_1 -space if and only if every subset is a Λ_s -set (or equivalently a V_s -set).

Proof. Let B be a subset of a semi- T_1 -space (X, τ) . Suppose that there exists a point $x \in X$ such that $x \notin B$. Then, $\{x\}^c$ is a semi-open set containing B . Then, by Definition 1 $B^{\Lambda_s} \subseteq \{x\}^c$. This implies $x \notin B^{\Lambda_s}$. Hence we have $B^{\Lambda_s} \subseteq B$ and $B^{\Lambda_s} = B$ (Proposition 3.1 (a)).

For the converse, if $x \in X$, then $X \setminus \{x\}^c$ is due to assumption a Λ_s -set. Hence, its complement $\{x\}$ is union of semi-closed sets and thus semi-closed. This shows that X is a semi- T_1 -space. \square

Corollary 3.8 *Every semi- T_1 -space is a semi- R_0 -space.*

Proof. The definition of semi- R_0 -spaces requires that every semi-open set is a V_s -set. \square

Example 3.9 Since indiscrete spaces (with at least two points) are semi- R_0 , then the separation axiom semi- R_0 is strictly below semi- T_1 . However, it is interesting to mention that even closed subspaces of semi- T_1 -spaces need not be semi- R_0 and that semi- T_1 -spaces need not be R_0 . Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Observe that X is semi- T_1 and that the closed subspace $A = \{a, c\}$ is not a semi- R_0 -space. Moreover, X is not R_0 .

Recall that a subset A is called *simply-open* if A is union of an open and a nowhere dense set. If $A \subseteq \text{Int}(\text{Cl}(A))$, then A is called *locally dense* or *preopen*. Sets which are dense in some regular closed subspace are called *semi-preopen* or β -sets.

Theorem 3.10 *For a topological space (X, τ) the following conditions are equivalent:*

- (1) X is a semi- T_1 -space;
- (2) Every locally dense (= preopen) subspace is a V_s -set;
- (3) Every β -open (= semi-preopen) subspace is a V_s -set.

Proof. (1) \Rightarrow (3) Follows from Proposition 3.7.

(3) \Rightarrow (2) Obvious, since every locally dense set is β -open.

(2) \Rightarrow (1) Let $x \in X$. It is well-known that every singleton is either locally dense or nowhere dense [7]. If $\{x\}$ is locally dense, then by (2), $\{x\}$ is union of semi-closed sets and hence semi-closed. If $\{x\}$ is nowhere dense, then it is clearly semi-closed. Thus every singleton of X is semi-closed and consequently X is a semi- T_1 -space. \square

Theorem 3.11 *For a topological space (X, τ) the following conditions are equivalent:*

- (1) *X is a semi- R_0 -space;*
- (2) *Every simply-open (= locally semi-closed) subspace is a V_s -set;*
- (3) *Every open subspace is a V_s -set.*

Proof. Since every open set is semi-open and since every semi-open set is simply-open, (1) \Rightarrow (3) and (2) \Rightarrow (1) are obvious.

(3) \Rightarrow (2) If $A \subseteq X$ is simply-open, then $A = U \cup N$, where $U \in \tau$ and N is nowhere dense. By (3), U is a V_s -set. Since every nowhere dense set is semi-closed, then by Proposition 3.5 A is a V_s -set. \square

Example 3.12 In the Euclidean plane (\mathbb{R}^2, τ) , every singleton $\{x\}$ is a Λ_s -set by Proposition 3.7. However, $\{x\}$ is not semi-open in (\mathbb{R}^2, τ) . Thus the converse of Proposition 3.1 (e) is not true in general.

4 $G.\Lambda_S$ -sets and $g.V_S$ -sets

In this section, by using the Λ_s -operator and V_s -operator, we introduce the classes of generalized Λ_s -sets (= $g.\Lambda_s$ -sets) and generalized V_s -sets (= $g.V_s$ -sets) as an analogy of the sets introduced by H. Maki [15].

Definition 3 In a topological space (X, τ) , a subset B is called a $g.\Lambda_s$ -set of (X, τ) if $B^{\Lambda_s} \subseteq F$ whenever $B \subseteq F$ and F is semi-closed.

Definition 4 In a topological space (X, τ) , a subset B is called a $g.V_s$ -set of (X, τ) if B^c is a $g.\Lambda_s$ -set of (X, τ) .

Remark 4.1 By D^{Λ_s} (resp. D^{V_s}) we will denote the family of all $g.\Lambda_s$ -sets (resp. $g.V_s$ -sets) of (X, τ) .

Proposition 4.2 *Let (X, τ) be a topological space. Then:*

- (a) *Every Λ_s -set is a $g.\Lambda_s$ -set.*
- (b) *Every V_s -set is a $g.V_s$ -set.*
- (c) *If $B_\lambda \in D^{\Lambda_s}$ for all $\lambda \in \Omega$, then $\bigcup_{\lambda \in \Omega} B_\lambda \in D^{\Lambda_s}$.*
- (d) *If $B_\lambda \in D^{V_s}$ for all $\lambda \in \Omega$, then $\bigcap_{\lambda \in \Omega} B_\lambda \in D^{V_s}$.*

Proof. (a) Follows from Definition 2 and Definition 3.

(b) Let B be a V_s -set subset of X . Then, $B = B^{V_s}$. By Proposition 3.1 (f) $(B^c)^{\Lambda_s} = (B^{V_s})^c = B^c$. Therefore, by (a) and Definition 4, B is a $g.V_s$ -set.

(c) Let $B_\lambda \in D^{\Lambda_s}$ for all $\lambda \in \Omega$. Then, by Proposition 3.1 (d) $[\bigcup_{\lambda \in \Omega} B_\lambda]^{\Lambda_s} = \bigcup_{\lambda \in \Omega} B_\lambda^{\Lambda_s}$. Hence, by hypothesis and Definition 3, $\bigcup_{\lambda \in \Omega} B_\lambda \in D^{\Lambda_s}$.

(d) Follows from (c) and Definition 4. \square

In general the intersection of two $g.\Lambda_s$ -sets is not a $g.\Lambda_s$ -set as shown by the following example.

Example 4.3 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. If $A = \{a, c\}$ and $B = \{b, c\}$ (as in [15, Example 3.3]). Then A and B are $g.\Lambda_s$ -sets, but $A \cap B = \{c\}$ is not a $g.\Lambda_s$ -set. We have: $D^{\Lambda_s} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $D^{V_s} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$.

The following example shows that the converse of Proposition 4.2 (a) (resp. (b)) is not true in general.

Example 4.4 Let (X, τ) be the space in Example 4.3. The subset $A = \{a, c\}$ is a $g.\Lambda_s$ -set but it is not a Λ_s -set.

Remark 4.5 By Remark 3.4 and Proposition 4.2 we have that:

- (i) If $A \in SO(X, \tau)$, then A is a $g.\Lambda_s$ -set;
- (ii) If $A \in SC(X, \tau)$, then A is a $g.V_s$ -set.

Proposition 4.6 *Let (X, τ) be a topological space.*

- (a) *For each $x \in X$, $\{x\}$ is a semi-open set or $\{x\}^c$ is a $g.\Lambda_s$ -set of (X, τ) .*
- (b) *For each $x \in X$, $\{x\}$ is a semi-open set or $\{x\}$ is a $g.V_s$ -set of (X, τ) .*

Proof. Suppose that $\{x\}$ is not semi-open. Then only semi-closed set F containing $\{x\}^c$ is X . Thus $(\{x\}^c)^{\Lambda_s} \subseteq F = X$ and $\{x\}^c$ is a $g.\Lambda_s$ -set of (X, τ) .

(b) Follows from (a) and Definition 3. \square

Corollary 4.7 *For a topological space (X, τ) , the Cantor-Bendixson derivative $D(X)$ is the set of all $g.V_s$ -singletons of (X, τ) .*

Proposition 4.8 *If A is a $g.\Lambda_s$ -set of a topological space (X, τ) and $A \subseteq B \subseteq A^{\Lambda_s}$, then B is a $g.\Lambda_s$ -set of (X, τ) .*

Proof. Since $A \subseteq B \subseteq A^{\Lambda_s}$, we have $A^{\Lambda_s} = B^{\Lambda_s}$ by Proposition 3.1 (b). Let F be any semi-closed subset of (X, τ) such that $B \subseteq F$. Then, we have $B^{\Lambda_s} = A^{\Lambda_s} \subseteq F$, since $A \subseteq B$ and A is $g.\Lambda_s$ -set. \square

In the following propositions we give a characterization of $g.V_s$ -sets (Definition 4) by using V_s -operations and we obtain results concerning such subsets.

Proposition 4.9 *A subset B of a topological space (X, τ) is a $g.V_s$ -set if and only if $U \subseteq B^{V_s}$ whenever $U \subseteq B$ and $U \in SO(X, \tau)$.*

Proof. Necessity. Let U be a semi-open subset of (X, τ) such that $U \subseteq B$. Then since U^c is semi-closed and $U^c \supseteq B^c$, we have $U^c \supseteq (B^c)^{\Lambda_s}$ by Definition 3 and Definition 4. Hence by Proposition 3.1 (f) $U^c \supseteq (B^{V_s})^c$. Thus, $U \subseteq B^{V_s}$.

Sufficiency. Let F be a semi-closed subset of (X, τ) such that $B^c \subseteq F$. Since F^c is semi-open and $F^c \subseteq B$, by assumption we have $F^c \subseteq B^{V_s}$. Then, $F \supseteq (B^{V_s})^c = (B^c)^{\Lambda_s}$ by Proposition 3.1 (f), and B^c is a $g.\Lambda_s$ -set, i.e., B is a $g.V_s$ -set. \square

As consequence of Proposition 4.9, we have:

Corollary 4.10 *Let B be a $g.V_s$ -set in a topological space (X, τ) . Then, for every semi-closed set F such that $B^{V_s} \cup B^c \subseteq F$, $F = X$ holds.*

Proof. The assumption $B^{V_s} \cup B^c \subseteq F$ implies $(B^{V_s})^c \cap B \supseteq F^c$. Since B is a $g.V_s$ -set, then by Proposition 4.9, we have $B^{V_s} \supseteq F^c$ and hence $(B^{V_s})^c \subseteq F$ and $\emptyset = (B^{V_s})^c \cap B^{V_s} \supseteq F^c$. Therefore, we have $X = F$. \square

Corollary 4.11 *Let B be a $g.V_s$ -set of (X, τ) . Then $B^{V_s} \cup B^c$ is a semi-closed set if and only if B is a V_s -set.*

Proof. Necessity. By Proposition 4.10, $B^{V_s} \cup B^c = X$. Thus $(B^{V_s})^c \cap B = \emptyset$. Hence, by Proposition 3.1 (g) $B = B^{V_s}$. *Sufficiency* is obvious. \square

Proposition 4.12 *Let B be a subset of topological space (X, τ) such that B^{V_s} is semi-closed. If $X = F$ holds for every semi-closed subset F such that $F \supseteq B^{V_s} \cup B^c$, then B is a $g.V_s$ -set.*

Proof. Let U be a semi-open subset contained in B . According to assumption, $B^{V_s} \cup U^c$ is semi-closed such that $B^{V_s} \cup B^c \subseteq B^{V_s} \cup U^c$. It follows that $B^{V_s} \cup U^c = X$ and hence $U \subseteq B^{V_s}$. By Proposition 4.9, B is a $g.V_s$ -set. \square

5 Characterization of semi- $T_{\frac{1}{2}}$ spaces

After the work of N. Levine [12] on semi-open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open sets instead of open sets. In this direction, P. Bhattacharyya and B. K. Lahiri [1] defined the concept of semi-generalized closed (= sg-closed) sets of a topological space in terms of semi-open sets. In recent years, the class of semi- $T_{\frac{1}{2}}$ -spaces has been of some interest, (i.e., the spaces where the classes of semi-closed sets and the sg-closed sets coincide). In this section we give a new characterization of semi- $T_{\frac{1}{2}}$ -spaces by using $g.V_s$ -sets. In order to achieve our purpose, we recall the following definitions (see also [1, 2, 15]).

Definition 5 A subset B of a topological space (X, τ) is said to be *semi-generalized closed set* (written briefly as *sg-closed*) [1] if $\text{sCl}(B) \subseteq O$ holds whenever $B \subseteq O$ and $O \in \text{SO}(X, \tau)$. Every semi-closed sets is sg-closed but the converse is not always true [1].

Definition 6 A topological space (X, τ) is said to be a *semi- $T_{\frac{1}{2}}$ -space* [1] if every *sg-closed* set in (X, τ) is semi-closed in (X, τ) .

In the following theorem we give a characterization of the class of semi- $T_{\frac{1}{2}}$ by using $g.V_s$ -sets.

Theorem 5.1 *Let (X, τ) be a topological space. Then the following statements are equivalent:*

- (a) (X, τ) is a semi- $T_{\frac{1}{2}}$ -space.
- (b) Every $g.V_s$ -set is a V_s -set.

Proof. (a) \Rightarrow (b): Suppose that there exists a $g.V_s$ -set B which is not a V_s -set. Since $B^{V_s} \subseteq B$ ($B^{V_s} \neq B$), then there exists a point $x \in B$ such that $x \notin B^{V_s}$. Then the singleton $\{x\}$ is not semi-closed. According to Proposition 4.3 of [16], $\{x\}^c$ is a *sg-closed* set. On the other hand, we have that $\{x\}$ is not semi-open (since B is a $g.V_s$ -set, $x \notin B^{V_s}$ and Proposition 4.9). Therefore, we have that $\{x\}^c$ is not semi-closed but it is a *sg-closed* set. This contradicts to the assumption that (X, τ) is a semi- $T_{\frac{1}{2}}$ -space.

(b) \Rightarrow (a): Suppose that (X, τ) is not a semi- $T_{\frac{1}{2}}$ -space. Then, there exists a *sg-closed* set B which is not semi-closed. Since B is not semi-closed, there exists a point x such that $x \notin B$ and $x \in \text{sCl}(B)$. By Proposition 4.6, we have the singleton $\{x\}$ is a semi-open set or it is a $g.V_s$ -set. When $\{x\}$ is semi-open, we have $\{x\} \cap B \neq \emptyset$ because $x \in \text{sCl}(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is a $g.V_s$ -set. If $\{x\}$ is not semi-closed, we have $\{x\}^{V_s} = \emptyset$ and hence $\{x\}$ is not a V_s -set. This contradicts to (b). Next, if $\{x\}$ is semi-closed, we have $\{x\}^c \supseteq \text{sCl}(B)$ (i.e., $x \notin \text{sCl}(B)$). In fact, the semi-open set $\{x\}^c$ contains the set B which is a *sg-closed* set. Then, this also contradicts to the fact that $x \in \text{sCl}(B)$. Therefore (X, τ) is a semi- $T_{\frac{1}{2}}$ -space. \square

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